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ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS WITH SHEAR DEFORMATIO--ETC(U)
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# ANALYSIS OF ORTHOTROPIC, LAMINATED SHELLS WITH SHEAR DEFORMATIONS

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U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35809

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A curved finite element is developed for the analysis of a laminated, axisymmetric shell of revolution with transverse shear deformation. The element stiffness relations are obtained from the principle of minimum potential energy.

The shell displacements represent four degrees of freedom at a node, two translation, two rotation.


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## CONTENTS

## I. INTRODUCTION

The continuing search for lighter, stronger, more economical missile structures has led to the investigation of fiber reinforced composites as a possible applicable type of construction. The inherent properties of this material pose many difficult problems for the designer and analyst. Tests on actual structures have proven that this material is unlike conventional structural materials in many respects. Although much has been learned about the material behavior in the past few years, there are still many areas that are unknown and unpredictable.

An initial effort [1] used a straight line element for a shell of revolution and included shear deformations. This gave very good results for single-layered shells.

Nickell and Sato [2] used a curved shell element but did not consider transverse shear deformations. A single layered shell, using a curved shell element and considering transverse shear, has been completed [3]. Several selected numerical examples were solved and the solutions are in good agreement with known results.

In this effort, the finite element method is used with a curved shell element considering a laminated shell of revolution and transverse shear deformations. The field equations similar to Reissner's theory of thick plates [4] are used as a guideline for formulating the shear deformation degree of freedom. The procedure employed is similar to that of Clough and Felippa [5]. The classical Kirchoff-Love assumption for normals to the midsurface is relaxed in favor of the assumed shear deformation mode.

## II. ELEMENT GEOMETRY

The shell to be considered is axisymmetric; therefore, it is sufficient to define only the shape of its meridional curve. The finite element method will be used for this analysis. The element is shown in Figure 1.

For this analysis, x - y are the local rectilinear coordinates, and r - z are the global coordinates. The angles shown in Figure 1 are related by the relation

$$\phi + \psi + \beta = \frac{\pi}{2} \quad . \tag{1}$$

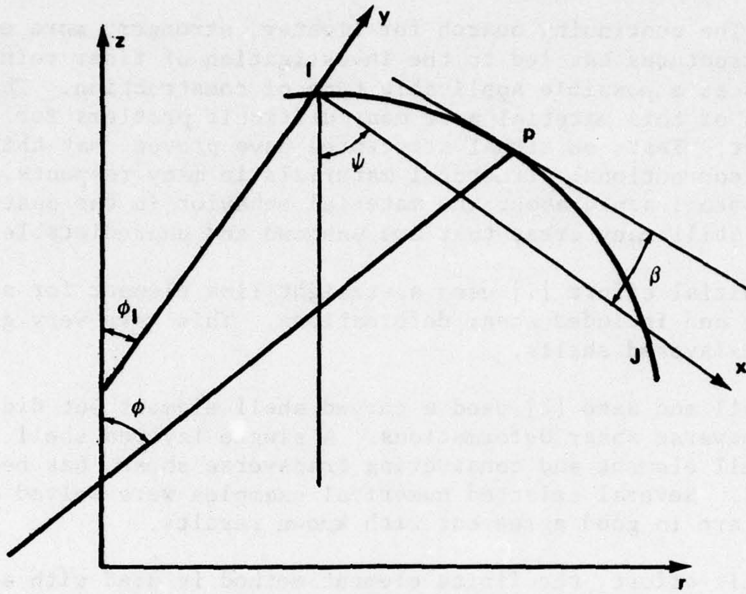


Figure 1. Curved shell element.

The angle $\phi$ is the angle between the normal to the reference surface and the axis of revolution. The angle $\psi$ is the angle between the chord of the element and the z-axis. The angle $\beta$ is defined to be the angle between the chord line (the x-axis) and the tangent to the curved surface.

From Equation (1),

$$\sin \beta = \cos(\phi + \psi) = \cos\phi \cos\psi - \sin\phi \sin\psi$$
$$\cos \beta = \sin(\phi + \psi) = \sin\phi \cos\psi + \cos\phi \sin\psi \quad . \qquad (2)$$

To approximate the meridional curve, the following substitute curve is assumed:

$$y = x\left(1 - \frac{x}{\ell}\right)\left(a_1 + a_2 \frac{x}{\ell} + a_3 \frac{x^2}{\ell^2} + a_4 \frac{x^3}{\ell^3}\right) \qquad (3)$$

where $\ell$ = chord length of the element. $(0 < x < \ell)$

Differentiating Equation (3) with respect to x

$$\frac{dy}{dx} = a_1 + \frac{2(a_2 - a_1)}{\ell} x + \frac{3(a_3 - a_2)}{\ell^2} x^2 + \frac{4(a_4 - a_3)}{\ell^3} x^3 - \frac{5a_4}{\ell^4} x^4$$

$$\frac{d^2y}{dx^2} = \frac{2(a_2 - a_1)}{\ell} + \frac{6(a_3 - a_2)}{\ell^2} x + \frac{12(a_4 - a_3)}{\ell^3} x^2 - \frac{20a_4}{\ell^4} x^3 \quad . \tag{4}$$

The constants $a_1$, $a_2$, $a_3$, and $a_4$ can be determined by evaluating Equations (4) at the end points

$$a_1 = \tan \beta_1$$

$$a_2 = \tan \beta_1 - \frac{\ell}{2R_1} \sec^3 \beta_1$$

$$a_3 = \frac{-\ell}{2R_2} \sec^3 \beta_2 + \frac{\ell}{R_1} \sec^3 \beta_1 - 4 \tan \beta_2 - 5 \tan \beta_1 \tag{5}$$

$$a_4 = \frac{\ell}{2R_2} \sec^3 \beta_2 - \frac{\ell}{R_1} \sec^3 \beta_1 + 3(\tan \beta_1 + \tan \beta_2) \, .$$

The following geometrical relations are given with respect to the element:

$$\Delta r = r_2 - r_1$$

$$\Delta z = z_1 - z_2$$

$$\ell = \sqrt{(\Delta r)^2 + (\Delta z)^2} \tag{6}$$

$$\sin \psi = \frac{\Delta r}{\ell}$$

$$\cos \psi = \frac{\Delta z}{\ell} \quad .$$

After the substitute curve has been established, all the geometric quantities can be written as follows:

$$\tan \beta = \frac{dy}{dx}$$

$$r = r_1 + x \sin \psi + y \cos \psi$$

$$\frac{dr}{dx} = \sin\psi + \tan\beta\cos\psi \tag{7}$$

$$\frac{d}{dS} = \cos\beta\frac{d}{dx}$$

$$\frac{d\beta}{dx} = \frac{d\beta}{dS}\frac{dS}{dx} = -\frac{1}{R}\sec\beta \quad .$$

Since

$$\frac{dy}{dx} = \tan\beta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\beta) = \sec^2\beta\frac{d\beta}{dx} \quad ,$$

therefore

$$\frac{d\beta}{dx} = -\frac{1}{R}\sec\beta \quad .$$

The quantity $d\beta/dS$ is negative since $\beta$ is decreasing as $S$ is increasing.

Therefore

$$\frac{d^2y}{dx^2} = -\frac{1}{R}\sec^3\beta \tag{8}$$

and

$$\cos\phi = \sin\beta\cos\psi + \cos\beta\sin\psi$$

$$\sin\phi = \cos\beta\cos\psi - \sin\beta\sin\psi \quad . \tag{9}$$

## III. TRANSFORMATION OF COORDINATE SYSTEMS

The displacement vector of a material point on the midsurface in the local principal curvilinear shell coordinate is denoted by

$$\{f_c\}^T = [u_c, w_c, \chi_c, \gamma_c] \tag{10}$$

where

$u_c$ = the displacement along the meridian.

$w_c$ = the transverse (normal) displacement.

$\chi_c$ = the rotation about a meridional tangent.

$\gamma_c$ = shear deformation.

The displacement components which refer to the local rectilinear coordinates, x-y, are

$$\{f_r\}^T = [u_r, w_r, \chi_r, \gamma_r] \tag{11}$$

and to the global coordinates, R-Z, are

$$\{f\}^T = [u, w, \chi, \gamma] \quad . \tag{12}$$

The transformation between these components can be seen as follows:

$$\begin{aligned} \{f_c\} &= [q_c]\ \{f_r\} \\ \{f_r\} &= [q_r]\ \{f\} \end{aligned} \tag{13}$$

where

$$[q_c] = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and

$$[q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad . \tag{15}$$

## IV. STRAIN-DISPLACEMENT RELATIONS

The strain-displacement relations can be written as

$$
\begin{aligned}
e_1 &= \frac{du_c}{dS} + \frac{w_c}{R} \\
e_2 &= \frac{1}{r}\left(u_c \cos\phi + w_c \sin\phi\right) \\
\kappa_1 &= -\frac{d}{dS}\left(\frac{dw_c}{dS} - \frac{u_c}{R} + \gamma\right) \\
\kappa_2 &= -\frac{\cos\phi}{r}\left(\frac{dw_c}{dS} - \frac{u_c}{R}\right) \\
\gamma_{1y} &= -\gamma \quad .
\end{aligned}
\tag{16}
$$

The strains defined in Equations (16) are now transformed into the local rectilinear coordinates as follows (recall that $dS = \frac{dx}{\cos\beta}$ and $\beta = \beta_1 - \frac{S}{R}$):

$$
\begin{aligned}
e_1 &= \frac{du_r}{dx} \cos^2\beta + \frac{dw_r}{dx} \cos\beta \sin\beta \\
e_2 &= \frac{1}{r}\left(u_r \sin\psi + w_r \cos\psi\right) \\
\kappa_1 &= -\cos^3\beta \frac{d^2 w_r}{dx^2} - \frac{2\cos\beta\sin\beta}{R_1}\frac{dw_r}{dx} + \sin\beta\cos^2\beta\frac{d^2u_r}{dx^2} \\
&\quad + \frac{\sin^2\beta - \cos^2\beta}{R_1}\frac{du_r}{dx} - \frac{d\gamma_r}{dx}\cos\beta \\
\kappa_2 &= -\frac{\cos\phi}{r}\left(\cos^2\beta\frac{dw_r}{dx} - \sin\beta\cos\beta\frac{du_r}{dx}\right) - \frac{\cos\phi}{r}\gamma_r \quad . \\
\gamma_{1y} &= -\gamma_r \quad .
\end{aligned}
\tag{17}
$$

## V. SHELL DISPLACEMENTS

$$u_c = u_r \cos\beta + w_r \sin\beta$$

$$w_c = -u_r \sin\beta + w_r \cos\beta$$

$$\frac{dw_c}{dS} = \left(\frac{dw_c}{dx}\right)\frac{dx}{dS} \tag{18}$$

$$= \frac{dx}{dS}\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta - u_r\cos\beta\frac{d\beta}{dx} - w_r\sin\beta\frac{d\beta}{dx}\right)$$

$$= \cos\beta\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta + \frac{u_r}{r}\right.$$

$$\left.+ \frac{w_r}{r}\tan\beta\right) .$$

The displacement field is assumed to be represented by

$$u_r = \alpha_1 + \alpha_2 x$$

$$w_r = \alpha_3 + \alpha_4 x + \alpha_5 x^2 + \alpha_6 x^3$$

$$\chi = \frac{dw_c}{dS} - \frac{u_c}{r} \tag{19}$$

$$= \cos\beta\left(-\frac{du_r}{dx}\sin\beta + \frac{dw_r}{dx}\cos\beta + \frac{u_r}{r} + \frac{w_r}{r}\tan\beta\right)$$

$$-\frac{u_r\cos\beta}{r} - \frac{w_r\sin\beta}{r}$$

$$= -\frac{du_r}{dx}\sin\beta\cos\beta + \frac{dw_r}{dx}\cos^2\beta$$

$$= -\alpha_2\sin\beta\cos\beta + (\alpha_4 + 2\alpha_5 x + 3\alpha_6 x^2)\cos^2\beta$$

$$\gamma_r = \alpha_7 + \alpha_8 x \quad .$$

In matrix notation this can be written as:

$$\begin{Bmatrix} u_r \\ w_r \\ \chi \\ \gamma \end{Bmatrix} = \begin{bmatrix} 1 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & x^2 & x^3 & 0 & 0 \\ 0 & -sc & 0 & c^2 & 2xc^2 & 3x^2c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{20}$$

where

$s = \sin \beta$, $c = \cos \beta$ .

This can be written symbolically as

$$\{f_r\} = [\phi] \{\alpha\} \tag{21}$$

where $\{\alpha\}$ is the generalized *coordinates* vector for the curved shell element.

The shell displacements shown in Equation (20) represent 4 degrees of freedom at a node, two translation, two rotation. The 8 degrees of freedom connected with the nodes of the element are written as the displacement vector

$$\{\delta_r^e\}^T = [u_{r1}, w_{r1}, \chi_{r1}, \gamma_{r1}, u_{r2}, w_{r2}, \chi_{r2}, \gamma_{r2}] \quad . \tag{22}$$

The generalized displacements $\{\alpha\}$ are related to the nodal point displacement vector $\{\delta^e\}$ by

$$\{\alpha\} = [A_r] \{\delta_r^e\} \quad . \tag{23}$$

$\{\alpha\}$ is evaluated as follows:

1) At $x = 0$

$\alpha_1 = u_{r1}$

$\alpha_3 = w_{r1}$

$\alpha_7 = \gamma_{r1}$

$-\alpha_2 \sin \beta_1 \cos \beta_1 + \alpha_4 \cos^2 \beta_1 = \chi_{r1}$ .

2) At $x = \ell$

$$\alpha_1 + \alpha_2 \ell = u_{r2}$$

$$\alpha_3 + \alpha_4 \ell + \alpha_5 \ell^2 + \alpha_6 \ell^3 = w_{r2}$$

$$\alpha_7 + \alpha_8 \ell = \gamma_{r2}$$

$$- \alpha_2 \sin \beta_2 \cos \beta_2 + \alpha_4 \cos^2 \beta_2 + 2\alpha_5 \ell \cos^2 \beta_2 + 3\alpha_6 \ell^2 \cos^2 \beta_2 = \chi_{r2} .$$

Solving this system of equations gives

$$\alpha_1 = u_{r1}$$

$$\alpha_2 = \frac{u_{r2} - u_{r1}}{\ell}$$

$$\alpha_3 = w_{r1}$$

$$\alpha_4 = \frac{\chi_{r1} + \sin \beta_1 \cos \beta_1 \left(\frac{u_{r2} - u_{r1}}{\ell}\right)}{\cos^2 \beta_1}$$

$$\alpha_5 = u_{r1} \left(\frac{\tan \beta_2 + 2 \tan \beta_1}{\ell^2}\right) - u_{r2} \left(\frac{\tan \beta_2 + 2 \tan \beta_1}{\ell^2}\right)$$

$$- \frac{3}{\ell^2} w_{r1} + \frac{3}{\ell^2} w_{r2} - \chi_{r1} \left(\frac{2}{\ell \cos^2 \beta_1}\right) - \chi_{r2} \left(\frac{1}{\ell \cos^2 \beta_2}\right)$$

$$\alpha_6 = - \frac{u_{r1} (\tan \beta_2 + \tan \beta_1)}{\ell^3} + \frac{u_{r2} (\tan \beta_2 + \tan \beta_1)}{\ell^3}$$

$$+ \frac{2}{\ell^3} w_{r1} - \frac{2}{\ell^3} w_{r2} + \chi_{r1} \left(\frac{1}{\ell^2 \cos^2 \beta_1}\right) + \chi_{r2} \left(\frac{1}{\ell^2 \cos^2 \beta_2}\right)$$

$$\alpha_7 = \gamma_{r1}$$

$$\alpha_8 = \frac{\gamma_{r2} - \gamma_{r1}}{\ell} .$$

Therefore, from (23)

$$[A_r] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\ell} & 0 & 0 & 0 & \frac{1}{\ell} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & 0 & b_1 & 0 & a_1 & 0 & 0 & 0 \\ a_2 & -\frac{3}{\ell^2} & -2b_2 & 0 & -a_2 & \frac{3}{\ell^2} & -b_4 & 0 \\ -a_3 & \frac{2}{\ell^3} & b_3 & 0 & a_3 & -\frac{2}{\ell^3} & b_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\ell} & 0 & 0 & 0 & \frac{1}{\ell} \end{bmatrix} \tag{24}$$

where

$$a_1 = \frac{\tan \beta_1}{\ell}$$

$$a_2 = \frac{2 \tan \beta_1 + \tan \beta_2}{\ell^2}$$

$$a_3 = \frac{\tan \beta_1 + \tan \beta_2}{\ell^3}$$

$$b_1 = \frac{1}{\cos^2 \beta_1}$$

$$b_2 = \frac{1}{\ell \cos^2 \beta_1}$$

$$b_3 = \frac{1}{\ell^2 \cos^2 \beta_1}$$

$$b_4 = \frac{1}{\ell \cos^2 \beta_2}$$

$$b_5 = \frac{1}{\ell \cos^2 \beta_2} .$$

The transformation of $\{\delta_r\}^e$ to the global coordinate $\{\delta\}^e$ is given by

$$\{\delta_r\}^e = [R] \{\delta\}^e \tag{25a}$$

where

$$\{\delta\}^e = \begin{Bmatrix} u_1 \\ w_1 \\ \chi_1 \\ \gamma_1 \\ u_2 \\ w_2 \\ \chi_2 \\ \gamma_2 \end{Bmatrix} \tag{25b}$$

and

$$[R] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\psi & -\cos\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{26}$$

Substituting (25) into (23) gives

$$\{\alpha\} = [A_r]\,\{\delta_r\}^e = [A_r]\,[R]\,\{\delta\}^e = [A]\,\{\delta\}^e \tag{27}$$

where

$$[A] = [A_r]\,[R]$$

$$[A] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-\sin\psi}{\ell} & \dfrac{\cos\psi}{\ell} & 0 & 0 & \dfrac{\sin\psi}{\ell} & \dfrac{-\cos\psi}{\ell} & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1\sin\psi & a_1\cos\psi & b_1 & 0 & a_1\sin\psi & -a_1\cos\psi & 0 & 0 \\ \begin{matrix} a_2\sin\psi \\ \dfrac{-3\cos\psi}{\ell^2}\end{matrix} & \begin{matrix} -a_2\cos\psi \\ \dfrac{-3\sin\psi}{\ell^2}\end{matrix} & -2b_2 & 0 & \begin{matrix} -a_2\sin\psi \\ \dfrac{+3\cos\psi}{\ell^2}\end{matrix} & \begin{matrix} a_2\cos\psi \\ \dfrac{+3\sin\psi}{\ell^2}\end{matrix} & -b_4 & 0 \\ \begin{matrix} -a_3\sin\psi \\ \dfrac{+2\cos\psi}{\ell^3}\end{matrix} & \begin{matrix} a_3\cos\psi \\ \dfrac{+2\sin\psi}{\ell^3}\end{matrix} & b_3 & 0 & \begin{matrix} a_3\sin\psi \\ \dfrac{-2\cos\psi}{\ell^3}\end{matrix} & \begin{matrix} -a_3\cos\psi \\ \dfrac{-2\sin\psi}{\ell^3}\end{matrix} & b_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\ell} & 0 & 0 & 0 & \dfrac{1}{\ell} \end{bmatrix} \tag{28}$$

## VI. STRESS-STRAIN RELATIONS

For an axisymmetric shell of revolution subjected to axisymmetric loadings, the stress resultants and couples can be expressed as

$$\begin{Bmatrix} N_1 \\ N_2 \\ M_1 \\ M_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 \\ E_{21} & E_{22} & E_{23} & E_{24} & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & 0 \\ E_{41} & E_{42} & E_{43} & E_{44} & 0 \\ 0 & 0 & 0 & 0 & E_{55} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ \kappa_1 \\ \kappa_2 \\ -\gamma_1 \end{Bmatrix} . \tag{29}$$

The quantities are related to the principal curvilinear coordinate system with 1 as meridional direction and 2 as circumferential direction. Symbolically this can be written as

$$\{S\} = [E]\ \{\epsilon\} \tag{30}$$

where [E] is the elasticity matrix. The detail derivation of [E] is given in Appendix A.

Substituting (19) into (17) gives

$$\begin{aligned} \{\epsilon\} &= [\Phi']\ \{\alpha\} \\ &= [\Phi']\ [A]\ \{\delta\}^e = [B]\ \{\delta\}^e \end{aligned} \tag{31}$$

where

$$[B] = [\Phi']\ [A]$$

$$\frac{du_r}{dx} = \alpha_2$$

$$\frac{dw_r}{dx} = \alpha_4 + 2\alpha_5\ x + 3\alpha_6\ x^2$$

$$\frac{d\gamma_r}{dx} = \alpha_8$$

$$\frac{d^2 w_r}{dx^2} = 2\alpha_5 + 6\alpha_6\ x$$

$$\frac{d^2 u_r}{dx^2} = 0$$

$$e_1 = \alpha_2 \cos^2\beta + (\alpha_4 + 2\alpha_5\ x + 3\alpha_6\ x^2) \cos\beta \sin\beta$$

$$e_2 = \frac{1}{r}\left[(\alpha_1 + \alpha_2\ x) \sin\psi + (\alpha_3 + \alpha_4\ x + \alpha_5\ x^2 + \alpha_6\ x^3) \cos\psi\right]$$

$$\kappa_1 = -\cos^3\beta\ (2\alpha_5 + 6\alpha_6\ x) - \frac{2\cos\beta\sin\beta}{R_1}(\alpha_4 + 2\alpha_5\ x + 3\alpha_6\ x^2)$$

$$+ \sin\beta\ \cancel{\cos^2}\ \beta\ (0) + \frac{\sin^2\beta - \cos^2\beta}{R_1}(\alpha_2) - \alpha_8 \cos\beta$$

$$\kappa_2 = -\frac{\cos\phi}{r}\left[\cos^2\beta\,(\alpha_4 + 2\alpha_5\,x + 3\alpha_6\,x^2) - \sin\beta\cos\beta\,\alpha_2\right]$$

$$-\frac{\cos\phi}{r}(\alpha_7 + \alpha_8\,x)$$

$$\gamma_{1y} = -(\alpha_7 + \alpha_8\,x)$$

$$[\phi'] = \begin{bmatrix} 0 & \cos^2\beta & 0 & \cos\beta\sin\beta & 2x\cos\beta\sin\beta & 3x^2\sin\beta\cos\beta & 0 & 0 \\ \frac{\sin\phi}{r} & \frac{x\sin\phi}{r} & \frac{\cos\phi}{r} & \frac{x\cos\phi}{r} & \frac{x^2\cos\phi}{r} & \frac{x^3\cos\phi}{r} & 0 & 0 \\ 0 & \frac{\sin^2\beta - \cos^2\beta}{R_1} & 0 & \frac{-2\cos\beta\sin\beta}{R_1} & \begin{matrix}-2\cos^3\beta \\ \frac{-4x\cos\beta\sin\beta}{R_1}\end{matrix} & \begin{matrix}-6x\cos^3\beta \\ \frac{-6x^2\cos\beta\sin\beta}{R_1}\end{matrix} & 0 & -\cos\beta \\ 0 & \frac{\cos\phi\sin\beta\cos\beta}{r} & 0 & \frac{-\cos\phi\cos^2\beta}{r} & \frac{-2x\cos\phi\cos^2\beta}{r} & \frac{-3x^2\cos\phi\cos^2\beta}{r} & \frac{-\cos\phi}{r} & \frac{-x\cos\phi}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -x \end{bmatrix} \tag{32}$$

From Equations (13) and (21)

$$\{f_r\} = [\phi]\,\{\alpha\} = [\phi]\,[A]\,\{\delta\}^e$$

$$\{f\} = [q_r]^{-1}\,\{f_r\} = [q_r]^T\,\{f_r\}$$

$$\{f\} = [q_r]^T\,[\phi]\,[A]\,\{\delta\}^e \tag{33}$$

$$= [N]\,\{\delta\}^e$$

where

$$[N] = [q_r]^T\,[\phi]\,[A]\quad . \tag{34}$$

The element stiffness matrix and equivalent nodal force may be obtained from the following formulas:

$$[k^e] = \iint_{A_e} [B]^T\,[E]\,[B]\,dA \tag{35}$$

$$\{F_p^e\} = \iint_{A_e} [N]^T\,\{P\}\,dA \tag{36}$$

where $\{P\}$ is the surface traction vector. The derivation of Equations (35) and (36) is given in Appendix B.

$$[\phi]^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -sc & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & c^2 & 0 \\ 0 & x^2 & 2\,x\,c^2 & 0 \\ 0 & x^3 & 3\,x^2\,c^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x \end{bmatrix} \tag{37}$$

$$[q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

$$[\phi]^T\,[q_r] = \begin{bmatrix} \sin\psi & -\cos\psi & 0 & 0 \\ x\sin\psi & -x\cos\psi & -sc & 0 \\ \cos\psi & \sin\psi & 0 & 0 \\ x\cos\psi & x\sin\psi & c^2 & 0 \\ x^2\cos\psi & x^2\sin\psi & 2\,x\,c^2 & 0 \\ x^3\cos\psi & x^3\sin\psi & 3\,x^2\,c^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x \end{bmatrix} \tag{39}$$

$$\{p_r\} = \begin{Bmatrix} p_x \\ p_y \\ 0 \end{Bmatrix} \tag{40}$$

where

$$p_x = p_t \cos\beta - p_n \sin\beta$$

$$p_y = p_t \sin\beta + p_n \cos\beta \quad .$$

$$\{p_r\} = [q_r]\,\{p\} \tag{41}$$

$$\{p\} = [q_r]^T\,\{p_r\} \tag{42}$$

$$= \begin{bmatrix} \sin\psi & \cos\psi & 0 & 0 \\ -\cos\psi & \sin\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ 0 \\ 0 \end{Bmatrix} \tag{43}$$

$$= \begin{Bmatrix} p_x \sin\psi + p_y \cos\psi \\ -p_x \cos\psi + p_y \sin\psi \\ 0 \\ 0 \end{Bmatrix} \tag{44}$$

$$
[\;\;]^T \, [q_r] \, \{p\} = \left\{ \begin{array}{l}
p_x \sin^2 \psi + p_y \cos \psi \sin \psi + p_x \cos^2 \psi - p_y \sin \psi \cos \psi \\
p_x \, x \sin^2 \psi + p_y \, x \cos \psi \sin \psi + p_x \, x \cos^2 \psi - p_y \, x \cos \psi \sin \psi \\
p_x \cos \psi \sin \psi + p_y \cos^2 \psi - p_x \cos \psi \sin \psi + p_y \sin^2 \psi \\
p_x \, x \cos \psi \sin \psi + p_y \, x \cos^2 \psi - p_x \, x \sin \psi \cos \psi + p_y \sin^2 \psi \\
p_x \, x^2 \sin \psi \cos \psi + p_y \, x^2 \cos^2 \psi - p_x \, x^2 \sin \psi \cos \psi + p_y \, x^2 \sin^2 \psi \\
p_x \, x^3 \sin \psi \cos \psi + p_y \, x^3 \cos^2 \psi - p_x \, x^3 \sin \psi \cos \psi + p_y \, x^3 \sin^2 \psi \\
0 \\
0
\end{array} \right\} \tag{45}
$$

$$
= \left\{ \begin{array}{l}
p_x \\
p_x \, x \\
p_y \\
p_y \, x \\
p_y \, x^2 \\
p_y \, x^3 \\
0 \\
0
\end{array} \right\} . \tag{46}
$$

## VII. RESULTS AND CONCLUSIONS

Various appropriate structures were analyzed by the computer code for this development. Comparisons were made with known solutions for single layered structures. The results were extremely close. Effort is now being made to find some results for laminated structures so that a comparison can be made.

The accuracy obtained by this method depends directly on the extent to which the assumed displacement patterns are able to reproduce the deformation actually developed within the element. Since the chosen displacement patterns satisfy the requirements of completeness and conformity (continuity of displacement at element boundary) as the size of the element decreases indefinitely, the solution obtained from the minimization of potential energy converges to the exact solution.

There still remains a need to add geometric and material nonlinearities to this analysis. The material quickly becomes nonlinear as the matrix material begins to crack or "craze" while the fibers are still intact. Also the capability for buckling prediction is one of the major needs for this type construction. These items are now being investigated and will be the subject of a later report.

## Appendix A.
## ELASTICITY MATRIX

Individual curved finite elements can, in general, be composed of a number of anisotropic layers of varying thickness along the meridional coordinate. For a single lamina, considering shear deformations, the constitutive relation is given as

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \\ \tau_{L\zeta} \\ \tau_{T\zeta} \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & 0 & 0 & 0 \\ Q'_{12} & Q'_{22} & 0 & 0 & 0 \\ 0 & 0 & Q'_{44} & 0 & 0 \\ 0 & 0 & 0 & Q'_{55} & 0 \\ 0 & 0 & 0 & 0 & Q'_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \\ \gamma_{L\zeta} \\ \gamma_{T\zeta} \end{Bmatrix} \tag{A-1}$$

where the transverse normal stress $\sigma_\zeta$ has been omitted and the laminae are orthotropic with respect to the principal elastic axes L-T. These axes need not coincide with the axes of the curvilinear coordinate system 1-2, (Figure A-1), (1 is the meridional direction) and

$$\begin{aligned} Q'_{11} &= E_L/(1 - \nu_{LT}\,\nu_{TL}) \\ Q'_{12} &= \nu_{LT}\,E_T/\,(1 - \nu_{LT}\,\nu_{TL}) \\ &= \nu_{TL}\,E_L/\,(1 - \nu_{TL}\,\nu_{LT}) \\ Q'_{22} &= E_T/(1 - \nu_{LT}\,\nu_{TL}) \\ Q'_{44} &= G_{LT} \\ Q'_{55} &= G_{L\zeta} \\ Q'_{66} & \quad G_{T\zeta} \end{aligned} \tag{A-2}$$

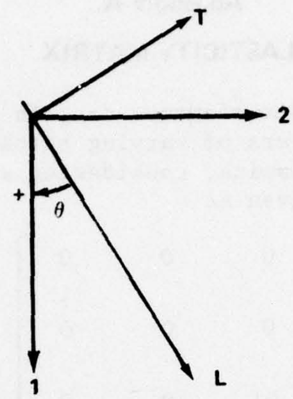


Figure A-1. Material axes.

Equation (A-1) can also be written for the Kth layer in the following forms:

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_k = \begin{bmatrix} Q'_{11} & Q'_{12} & 0 \\ Q'_{12} & Q'_{22} & 0 \\ 0 & 0 & Q'_{44} \end{bmatrix}_k \begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{Bmatrix} \quad \text{(A-3)}$$

and

$$\begin{Bmatrix} \tau_{L\zeta} \\ \tau_{T\zeta} \end{Bmatrix}_k = \begin{bmatrix} Q'_{55} & 0 \\ 0 & Q'_{66} \end{bmatrix}_k \begin{Bmatrix} \gamma_{L\zeta} \\ \gamma_{T\zeta} \end{Bmatrix}_k .$$

To develop a theory for structural laminates with individual layers having their elastic axes oriented at various angles relative to the coordinate axes, the stress-strain Equations (A-3) must be rotated through the positive angle $\theta$ so that the transformed stress-strain equations are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{14} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{24} \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{44} \end{bmatrix}_k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}_k \quad \text{(A-4)}$$

and

$$\begin{Bmatrix} \tau_{1\zeta} \\ \tau_{2\zeta} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{55} & 0 \\ 0 & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix}_k$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q'_{11} \cos^4 \theta + 2(Q'_{12} + 2\,Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{22} \sin^4 \theta \\
\bar{Q}_{12} &= (Q'_{11} + Q'_{22} - 4\,Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{12} (\sin^4 \theta + \cos^4 \theta) \\
\bar{Q}_{22} &= Q'_{11} \sin^4 \theta + 2(Q'_{12} + 2\,Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{22} \cos^4 \theta \qquad \text{(A-5)} \\
\bar{Q}_{14} &= (Q'_{11} + Q'_{12} - 2\,Q'_{44}) \sin \theta \cos^3 \theta + (Q'_{12} - Q'_{22} + 2\,Q'_{44}) \sin^3 \theta \cos \theta \\
\bar{Q}_{24} &= (Q'_{11} - Q'_{12} - 2\,Q'_{44}) \sin^3 \theta \cos \theta + (Q'_{12} - Q'_{22} + 2\,Q'_{44}) \sin \theta \cos^3 \theta \\
\bar{Q}_{44} &= (Q'_{11} + Q'_{22} - 2\,Q'_{12} - 2\,Q'_{44}) \sin^2 \theta \cos^2 \theta + Q'_{44} (\sin^4 \theta \cos^4 \theta) \\
\bar{Q}_{55} &= Q'_{55} \\
\bar{Q}_{66} &= Q'_{66} \quad .
\end{aligned}$$

Substituting the midsurface strain and curvatures into Equations (A-4) the following expression is obtained:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [\bar{Q}]_k \begin{Bmatrix} e_1 \\ e_2 \\ 2e_{12} \end{Bmatrix} + \zeta\, [\bar{Q}]_k \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ 2\kappa_{12} \end{Bmatrix} \qquad \text{(A-6)}$$

and

$$\begin{Bmatrix} \tau_{1\zeta} \\ \tau_{2\zeta} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55} & 0 \\ 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix}$$

By integrating over the total thickness of the laminate, the generalized stress resultants in terms of midsurface strain and curvature are given as

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_{12} \\ Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} [C] & [D^*] & 0 \\ [D^*] & [D] & 0 \\ 0 & 0 & [S] \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ 2e_{12} \\ \kappa_1 \\ \kappa_2 \\ 2\kappa_{12} \\ \gamma_{1\zeta} \\ \gamma_{2\zeta} \end{Bmatrix} \tag{A-7}$$

where

$$[C] = \sum_{k=1}^{m} [\bar{Q}^{(k)}] (h_k - h_{k-1})$$

$$[D^*] = \frac{1}{2} \sum_{k=1}^{m} [\bar{Q}^{(k)}] (h_k^2 - h_{k-1}^2)$$

$$[D] = \frac{1}{3} \sum_{k=1}^{m} [\bar{Q}^{(k)}] (h_k^3 - h_{k-1}^3) \tag{A-8}$$

$$[S] = \sum_{k=1}^{m} [\bar{Q}^{(k)}] (h_k - h_{k-1})$$

in which $h_k$ and $h_{k-1}$ = the distances, respectively, from the midsurface to the inner and outer surfaces of the k-th layer.

For an axisymmetric shell of revolution subjected to axisymmetric loadings, $N_{12} = M_{12} = Q_2 = e_{12} = \kappa_{12} = \gamma_{2\zeta} = 0$.

Hence,

$$\begin{Bmatrix} N_1 \\ N_2 \\ M_1 \\ M_2 \\ Q_1 \end{Bmatrix} = \begin{bmatrix} [C] & [D^*] & 0 \\ [D^*] & [D] & 0 \\ 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ \kappa_1 \\ \kappa_2 \\ \gamma_{1\zeta} \end{Bmatrix} \tag{A-9}$$

or symbolically

$$\{S\} = [E]\,\{\epsilon\} \quad . \tag{A-10}$$

## Appendix B.

## ELEMENT STIFFNESS MATRIX

The element stiffness matrix is found by writing the total potential energy of the axisymmetric shell of revolution and minimizing it for the imposed constraints and loading conditions.

The potential energy for a linear elastic shell of revolution in the absence of thermal and body forces can be formulated as follows:

$$\pi = \iiint_V \frac{1}{2} \{\epsilon\}^T \{\sigma\} \, dV - \int_{A_1} \{f\}^T \{P\} \, dA \qquad \text{(B-1)}$$

where the vectors $\{\epsilon\}$, $\{\sigma\}$, $\{f\}$, and $\{P\}$ represent the strain, stress, displacement, and equivalent surface traction vectors, respectively.

Introducing the stress resultant vector

$$\{S\} = t \{\sigma\} \qquad \text{(B-2)}$$

where t is the thickness of the shell, Equation (B-1) may be written as

$$\pi = \iiint_V \frac{1}{2} \{\epsilon\}^T \{S\} \frac{dV}{t} - \iint_{A_1} \{f\}^T \{P\} \, dA \quad . \qquad \text{(B-3)}$$

The first integral is evaluated over the entire volume V of the shell and the second over the portion $A_1$ of the midsurface of the shell, where the equivalent surface tractions are prescribed. Since the state of displacement throughout the shell is defined element by element, the total potential energy may be considered as the sum of the potential energies of all individual elements, i.e.,

$$\pi = \sum_e \pi^e \quad .$$

The potential energy contribution of element "e" will now be considered. The state of displacement defined for the element in local rectilinear coordinates x-y can be expressed in matrix form in Equation (21) as

$$\{f_r\} = [\phi] \{\alpha\} = [\phi][A_r] \{\delta_r^e\} \quad . \qquad \text{(B-4)}$$

Transformation of $\{f_r\}$ into the global coordinate system may be obtained from Equation (13)

$$\{f\} = [q_r]^T \{f_r\} = [N] \{\delta_r\}^e \tag{B-5}$$

where

$$[N] = [q_r]^T [\phi] [A_r] \tag{B-6}$$

and the column vector $\{\delta_r\}^e$ represents the eight discrete parameters (nodal point displacements) of the element as given in Equation (25b). The matrix [N] is a function of spatial coordinates and describes the defined displacement pattern.

Substituting Equation (27) into Equation (31) the following strain-displacement relations are obtained:

$$\{\epsilon\} = [B] \{\delta\}^e \tag{B-7}$$

where

$$[B] = [\phi'] [A] \quad . \tag{B-8}$$

Equation (B-8) is a matrix relating the nodal point displacement vector to the strain vector. The elastic stress-strain relations can be expressed as

$$\{S\} = [E] \{\epsilon\} \tag{B-9}$$

where [E] is a function of the elastic properties of the element. Each element can be assigned different elastic properties. If the relations in Equations (B-9), (B-5) and (B-7) are substituted into (B-3), the potential energy contribution for the element becomes

$$\pi^e = \iiint_{V_e} \frac{1}{2} \{\delta^e\}^T [B]^T [E] [B] \{\delta^e\} \frac{dV}{t} - \iint_{A_{1_e}} \{\delta^e\}^T [N]^T \{P\} \, dA \tag{B-10}$$

where $V_e$ is the volume of the element and $A_{1_e}$ is that part of the midsurface area of the element which coincides with the midsurface area $A_1$ of the shell over which the equivalent surface tractions are prescribed.

Since the discrete parameters $\{\delta^e\}$ are not a function of spatial coordinates, the potential energy of the element may be written as

$$\pi^e = \{\delta^e\}^T \left[ \iiint_{V_e} \frac{1}{2} [B]^T [E][B] \frac{dV}{t} \right] \{\delta^e\} - \{\delta^e\}^T \iint_{A_{1_e}} [N]^T \{P\} \, dA. \quad \text{(B-11)}$$

Since the assumed displacement patterns for each element satisfy various requirements such as completeness and conformity, the best values that can be obtained for the total nodal point displacements of the finite element representation of shells of revolution are those that minimize the total potential energy of the shell under the constraints imposed; i.e., the best value of $\{\delta\}$ are those that satisfy the system of linear equations

$$\frac{\partial \pi}{\partial \{\delta\}} = 0 \quad \text{(B-12)}$$

where $\{\delta\}$ is the total nodal displacement vector of the system.

In forming the system of Equations (B-12), it is convenient to have an expression for the spatial derivatives of the potential energy of each element "e" with respect to its own nodal point displacement vector $\{\delta^e\}$, i.e.,

$$\frac{\partial \pi^e}{\partial \{\delta^e\}} = \left[ \frac{\partial \pi^e}{\partial u_I} \; \frac{\partial \pi^e}{\partial w_I} \; \frac{\partial \pi^e}{\partial \chi_I} \; \frac{\partial \pi^e}{\partial \gamma_I} \; \frac{\partial \pi^e}{\partial u_J} \; \frac{\partial \pi^e}{\partial w_J} \; \frac{\partial \pi^e}{\partial \chi_J} \; \frac{\partial \pi^e}{\partial \gamma_J} \right] \quad . \quad \text{(B-13)}$$

By use of Equation (B-10), this expression can be obtained as

$$\frac{\partial \pi^e}{\partial \{\delta^e\}} = \left[ \iiint_{V_e} [B]^T [E][B] \frac{dV}{t} \right] \{\delta^e\} - \left[ \iint_{A_{1_e}} [N]^T \{P\} \, dA \right] \quad . \quad \text{(B-14)}$$

The terms in the first and second brackets are normally defined as the element stiffness matrix $[K^e]$ and the element generalized nodal point force $\{F^e\}$, respectively. Hence,

$$[K^e] = \iiint_{V_e} [B]^T [E][B] \frac{dV}{t} \quad \text{(B-15)}$$

$$\{F^e\} = \iint_{A_{1_e}} [N]^T \{P\} \, dA \quad . \quad \text{(B-16)}$$

By properly combining the submatrices in Equation (B-14) obtained for each element, the total matrix equation representing Equation (B-12) can be constructed as

$$[K]\ \{\delta\} = \{F\} \tag{B-17}$$

and then solved for the nodal point displacements. Once the nodal point displacements are obtained, the corresponding stress resultants, stresses, and strains for the defined displacement patterns can be calculated from Equations (B-7) and (B-9).

If Equation (B-8) is substituted into Equation (B-15) and the volume increment for a shell of revolution is taken as

$$dV = 2\pi t\ R(x)\ dx, \tag{B-18}$$

then the element stiffness matrix for the axisymmetric shell element takes the form

$$\begin{aligned} [K^e] &= 2\pi \int_0^{\ell} [B]^T\ [E]\ [B]\ \ R(x)\ dx \\ &= 2\pi\ [A]^T\ [G]\ [A] \end{aligned} \tag{B-19}$$

where

$$[G] = \int_0^{\ell} [\phi']^T\ [E]\ [\phi']\ R(x)\ dx \quad . \tag{B-20}$$

The integration is over the chord length of the meridian cross section of element.

It is assumed that the equivalent surface traction over the mid-surface area $A_1$ where tractions are prescribed varies linearly between the two nodal circles I and J. That is,

$$\{P_c\}^T = [0\ \ (P_n + P_n'\ x)\ \ 0\ \ \ 0] \tag{B-21}$$

where $\{P_c\}$ is the surface traction vector expressed in local curvilinear coordinates. Transforming into global coordinates the following is obtained:

$$\{P\} = [q_r]^T\ [q_c]^T\ \{P_c\} \quad . \tag{B-22}$$

Substituting Equations (B-6) and (B-22) into Equation (B-16) the generalized element nodal force vector becomes

$$\{F^e\} = 2\pi \int_0^{\ell} [A_r]^T \, [\phi]^T \, [q_r] \, [q_r]^T \, [q_c]^T \, \{P_c\} \, R(x) \, dx \tag{B-23}$$

or

$$\{F^e\} = 2\pi [A_r]^T \int_0^{\ell} [\phi]^T \, [q_c]^T \, \{P_c\} \, R(x) \, dx$$

where

$$[\phi]^T \, [q_c]^T \, \{P_c\} = P_n \begin{Bmatrix} -\sin\beta \\ -x \sin\beta \\ \cos\beta \\ x \cos\beta \\ x^2 \cos\beta \\ x^3 \cos\beta \\ 0 \\ 0 \end{Bmatrix} + P_n' \begin{Bmatrix} -x \sin\beta \\ -x^2 \sin\beta \\ x \cos\beta \\ x^2 \cos\beta \\ x^3 \cos\beta \\ x^4 \cos\beta \\ 0 \\ 0 \end{Bmatrix} .$$

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